SO(3)-MONOPOLES: THE OVERLAP PROBLEM

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Abstract. The SO(3)-monopole program, initiated by Pidstrigatch and Tyurin [27], yields a relationship between the Donaldson and Seiberg-Witten invariants through a cobordism between the moduli spaces defining these invariants. The main technical difficulty in this program lies in describing the links of singularities in this cobordism arising from the Seiberg-Witten moduli subspaces. In [4], we defined maps which, essentially, define normal bundles of strata of these singularities. The link in question is then the boundary of the union of the tubular neighborhoods associated with these normal bundles. However, the SO(3)-monopole program requires the computation of intersection numbers with links where more than one stratum appears in the family of singularities and thus more than one tubular neighborhood appears in the definition of the link. Computations of intersection numbers in unions of open sets have proved difficult for even two open sets, [25, 20]. In this note, we give a brief introduction to our article [6], in which we implement these computations.

1. Introduction

Before the introduction of Seiberg-Witten invariants [32], the Donaldson invariants were the chief means of distinguishing between smooth structures on four-manifolds. In [32], Witten not only defined Seiberg-Witten invariants, which are easier to compute, but also conjectured a relation between the Donaldson and Seiberg-Witten invariants. Assuming the conjecture, one see that the Donaldson and Seiberg-Witten invariants contain the same information about the smooth structure of a four-manifold.

In [27], Pidstrigatch and Tyurin introduced the SO(3)-monopole program to prove Witten's conjecture; see also an account by Okonek and Teleman in [24, 23]. We provide a description of this program in [8].

In [9, 10, 11, 12], we proved that for an appropriate choice of a spin^u structure $\mathfrak{t}$ (defined in §2.1), an Uhlenbeck-type compactification of the moduli space of SO(3) monopoles, $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$, defines a smoothly-stratified cobordism between a link of a moduli space of anti-self-dual connections and links of singularities of the form $M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X)$ where $M_{\mathfrak{s}}$ is the Seiberg-Witten moduli space associated to the spin^c structure $\mathfrak{s}$ (see [22]). In this note, we

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give an introduction to the ideas underlying the proof of our main result in [6]. This result relies on a technical result, stated here as Theorem 4.2, the proof of which should be a routine extension of the results of [4], and which will appear in [5].

Theorem 1.1. *Let X be a smooth, oriented manifold with $b^1(X) = 0$. Let $\mathrm{Char}(X) \subset H^2(X;\mathbb{Z})$ be the set of integral lifts of $w_2(X)$. Then, for $h \in H_2(X;\mathbb{R})$, $w \in H^2(X;\mathbb{Z})$, and generator $x \in H_0(X;\mathbb{Z})$,*

$$D_X^w(h^{\delta-2m}x^m) = -\sum_{c\in\mathrm{Char}(X)} SW_X(c)g^w_{X,\delta,m,c}(h^{\delta-2m}x^m), \tag{1.1}$$

where $D_X^w(h^{\delta-2m}x^m)$ is the Donaldson invariant of X, $SW_X(c)$ is the Seiberg-Witten invariant of the spin^c structure $\mathfrak{s}$ with $c_1(\mathfrak{s}) = c$, and

$$g^w_{X,\delta,m,c} : \mathrm{Sym}(H_0(X;\mathbb{Z}) \oplus H_2(X;\mathbb{Q})) \to \mathbb{Q}$$

is a universal function depending only on δ, m, w, c and the homotopy type of X.

The proof of Theorem 1.1 proceeds as follows. On a dense, open subspace

$$\mathcal{M}^{*,0}_{\mathfrak{t}}/S^1 \subset \bar{\mathcal{M}}_{\mathfrak{t}}/S^1,$$

there are geometric representatives $\bar{\mathcal{V}}(h^{\delta-2m}x^m)$ and $\bar{\mathcal{W}}$ which can be thought of, following [17, §2(ii)], as representatives of homology classes. The geometric representative $\bar{\mathcal{V}}(h^{\delta-2m}x^m)$ is essentially that defined in [17, §2(ii)], where it is used to compute the Donaldson invariant $D_X^w(h^{\delta-2m}x^m)$. The geometric representative $\bar{\mathcal{W}}$ is a representative of the homology class Poincaré dual to a multiple of the first Chern class of the S^1 action on $\mathcal{M}^{*,0}_{\mathfrak{t}}$. The intersection,

$$\mathcal{M}^{*,0}_{\mathfrak{t}}/S^1 \cap \bar{\mathcal{V}}(h^{\delta-2m}x^m) \cap \bar{\mathcal{W}}^{n-1},$$

where $n = n_a(\mathfrak{t})$ is a non-negative integer defined in equation (2.9), is a collection of oriented, one-dimensional manifolds. The cobordism given by these one-dimensional manifolds yields an identity:

$$2^{n-1}D_X^w(h^{\delta-2m}x^m) = -\sum_{\mathfrak{s}\in\mathrm{Spin}^c(X)} \#\left(\bar{\mathcal{V}}(h^{\delta-2m}x^m) \cap \bar{\mathcal{W}}^{n-1} \cap \mathbf{L}_{\mathfrak{t},\mathfrak{s}}\right), \tag{1.2}$$

where $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ is the link of $M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X)$ in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$. Theorem 1.1 then follows from a partial computation of the intersection numbers in (1.2):

$$\#\left(\bar{\mathcal{V}}(h^{\delta-2m}x^m) \cap \bar{\mathcal{W}}^{n-1} \cap \mathbf{L}_{\mathfrak{t},\mathfrak{s}}\right) = SW_X(\mathfrak{s})g^w_{X,\delta,m,c_1(\mathfrak{s})}(h^{\delta-2m}x^m), \tag{1.3}$$

where $g^w_{X,\delta,m,c_1(\mathfrak{s})}$ is as defined in Theorem 1.1 and $c_1(\mathfrak{s})$ is the characteristic class defined in (2.2). A more detailed statement of (1.3) appears in Theorem 2.1.

The proofs of Equation (1.3) for $\ell = 0$ and $\ell = 1$ appear in [11] and [12] respectively. Both of these proofs proceed by presenting the link $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ as the intersection of the zero-locus of an obstruction bundle with the boundary of a disk bundle (or an orbifold disk bundle when $\ell = 1$), showing that the

intersection number is given by a cohomological pairing, and then computing the cohomology ring of this disk bundle. For $\ell > 1$, the link $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ is the intersection of the zero-locus of an obstruction bundle with the boundary of a *union* of cone bundles. By a cone bundle, we mean a fiber bundle whose fiber is given by a cone on a space which need not be a sphere. The topology of these cones is sufficiently complicated that the implied computation in Equation (1.3) can only be partial and not as explicit as one might like. A more profound difficulty is that, as in all Mayer-Vietoris arguments, if one wants to compute the cohomology of a union, one must understand the intersection of the elements of that union. We refer to this problem as the *overlap problem*. Our goal in this note is to provide an exposition of the lengthy proof [6] of Equation (1.3), in which we solve the overlap problem.

1.1. **A model for intersection theory on stratified links.** As noted above, the link $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ is a subspace of the boundary of a union of cone bundles and to compute the intersection number (1.3), we must understand the intersections of these cone bundles. A precise accounting of what information about such a neighborhood is necessary for this computation can be given more concisely in an abstract language which we now introduce.

Let $S \subset Y$ be a closed subspace of a smoothly stratified space Y by which we mean that Y is the disjoint union of smooth manifolds Y_i, which we call *strata*, where i lies in a partially ordered set satisfying $i < j$ if and only if $Y_i \subseteq \operatorname{cl}(Y_j)$. We assume that Y is the closure of the highest stratum Y_n and refer to $\cup_{i<n} Y_i$ as the *lower strata*. In addition, we assume the strata of Y satisfy the *condition of frontier* that $Y_i \cap \operatorname{cl}(Y_j) \neq \varnothing$ only if $Y_i \subseteq \operatorname{cl}(Y_j)$. We will write $S_i = S \cap Y_i$ and assume that S_i is a smooth submanifold of Y_i.

If S were a submanifold of a smooth manifold, the standard language for computing intersection numbers with the link of S would be to introduce a normal bundle $N \to S$ and an embedding $\boldsymbol{\gamma} : \mathcal{O} \subset N \to Y$ where $\mathcal{O}$ is a closed disk subbundle. The link is then defined as $\mathbf{L} = \partial(\boldsymbol{\gamma}(\mathcal{O}))$.

Neighborhoods of stratified subspaces need not be so simple. Because stratified spaces are not manifolds, neighborhoods of subspaces cannot necessarily be parameterized by vector bundles. Instead of vector bundles, such neighborhoods are parameterized by cone bundles with fiber given by a cone on a more complicated topological space. Much of the following description applies to a wide variety of such spaces, including Whitney or Thom-Mather stratified spaces [31, pp. 2-4], [13, §1.5], [26, §1.4.1].

Definition 1.2. A subspace S of a stratified space Y has *smooth, local cone bundle neighborhoods* if the following holds. For each stratum S_i, there are

- A neighborhood $\mathcal{O}_i$ of S_i in Y with $\mathcal{O}_i \cap \mathcal{O}_j$ non-empty if and only if $i < j$ or $j < i$,
- A fiber bundle $\pi_i : N_i \to S_i$ with fiber F_i, a cone, wherein we identify S_i with the section of π_i given by the cone point,
- A homeomorphism $\boldsymbol{\gamma}_i$ from a neighborhood of S_i in N_i with $\mathcal{O}_i$.

Let $t_i : \mathcal{O}_i \to [0,\infty)$ be the function defined by the composition of $\boldsymbol{\gamma}_i^{-1}$ and the cone parameter on N_i. We will also write $\pi_i : \mathcal{O}_i \to S_i$ for the map $\pi_i \circ \boldsymbol{\gamma}_i^{-1}$. These maps satisfy the *Thom-Mather control conditions* if

- The map $(\pi_i, t_i) : \mathcal{O}_i \to S_i \times [0,\infty)$ is a smooth submersion on each stratum,
- For $i < j$, on $\mathcal{O}_i \cap \mathcal{O}_j$,

$$\pi_i \circ \pi_j = \pi_i, \quad t_i \circ \pi_j = t_i. \tag{1.4}$$

We say the control data $\{(\mathcal{O}_i, \pi_i, t_i)\}$ has *compatible structure groups* if

- The structure group of each bundle $\pi_i : N_i \to S_i$ is a compact Lie group G_i,
- For $i < j$, the intersection $\mathcal{O}_i \cap \mathcal{O}_j$ is a G_i-subbundle of N_i and a G_j-subbundle of N_j,
- For $i < j$, on the intersection $\mathcal{O}_i \cap \mathcal{O}_j$, the level sets $t_j^{-1}(\varepsilon)$ are G_i-subbundles.

For spaces satisfying Definition 1.2, we define the link of S in Y by,

$$\mathbf{L} = \partial(\cup_i \mathcal{O}_i).$$

If the intersection of $\mathbf{L}$ with the lower strata has codimension greater than or equal to two in $\mathbf{L}$, then $\mathbf{L}$ has a fundamental class $[\mathbf{L}]$. The intersection number of a geometric representative or divisor $\mathcal{V}$ with $\mathbf{L}$ can be represented, through a duality argument, as a cohomological pairing:

$$\#(\mathcal{V} \cap \mathbf{L}) = \langle \mu, [\mathbf{L}] \rangle,$$

with an appropriate cohomology class μ. To give a partial computation of this pairing, even when we do not know the topology of the fibers F_i, we decompose $\mathbf{L}$ as

$$\mathbf{L} = \cup_i \, \mathbf{L}_i, \quad \text{where} \quad \mathbf{L}_i = \boldsymbol{\gamma}_i \left(t_i^{-1}(\varepsilon_i) \right) - \cup_{j \neq i} \boldsymbol{\gamma}_j \left(t_j^{-1}[0, \varepsilon_j) \right). \tag{1.5}$$

For generic choices of the constants ε_i, the components $\mathbf{L}_i$ of this decomposition will be smoothly-stratified, closed, codimension-zero subspaces of $\mathbf{L}$ in which each stratum is a smooth manifold with corners (see [15, p. 7] or [21, Definition 1.2.2]). The boundary of each component $\mathbf{L}_i$ can be described as:

$$\partial \mathbf{L}_i = \cup_{j \neq i} \, \mathbf{L}_i \cap \mathbf{L}_j = \cup_{j \neq i} \, \mathbf{L}_i \cap t_j^{-1}(\varepsilon_j).$$

Because of the control on the overlaps $\mathcal{O}_i \cap \mathcal{O}_j$ given by the assumption on compatible structure groups, each component $\mathbf{L}_i$ is a G_i subbundle of $N_i \to S_i$ with fiber which we will write as $F_i(\boldsymbol{\varepsilon})$, appearing in the diagram

$$\begin{array}{ccc} \mathbf{L}_i & \xrightarrow{\tilde{f}_i} & \mathrm{EG}_i \times_{G_i} F_i(\boldsymbol{\varepsilon}) \\ {\scriptstyle \pi_i}\downarrow & & {\scriptstyle p_i}\downarrow \\ S_i & \xrightarrow{f_i} & \mathrm{BG}_i, \end{array} \tag{1.6}$$

Let μ_i be the restriction of μ to $\mathbf{L}_i$. Assume that μ_i is given by a product of classes pulled back from S_i and of G_i-equivariant cohomology classes on $F_i(\boldsymbol{\varepsilon})$. Then, the assumption on compatible structure groups allows us to chose a representative of μ such that the restrictions μ_i have compact support on $\mathbf{L}_i$ and are given by a product, $\mu_i = \pi_i^* x_i \smile \tilde{f}_i^* \nu_i$ where $x_i \in H_c^\bullet(S_i)$ and

$$\nu_i \in H^\bullet(\mathrm{EG}_i \times_{G_i} F(\boldsymbol{\varepsilon})))$$

has compact vertical support with respect to p_i. Then the decomposition of $\mathbf{L}$ in (1.5) yields the equalities

$$\begin{aligned}
\langle \mu, [\mathbf{L}] \rangle &= \sum_i \langle \mu, [\mathbf{L}_i] \rangle \\
&= \sum_i \langle (\pi_i)_* \mu_i, [S_i] \rangle \\
&= \sum_i \langle (\pi_i)_* (\pi_i^* x_i \smile \tilde{f}_i^* \nu_i), [S_i] \rangle \\
&= \sum_i \langle x_i \smile f_i^*((p_i)_* \nu_i), [S_i] \rangle.
\end{aligned} \tag{1.7}$$

The final step in (1.7) is known as a pushforward-pullback argument (see [28, Proposition 1.15]).

This argument allows us to isolate the topology of the fibers F_i in universal constants, producing the desired partial computation of the intersection number $\#(\mathcal{V} \cap \mathbf{L})$ in terms of the homotopy class of the classifying map $f_i : S_i \to \mathrm{BG}_i$ without explicit knowledge of the fibers F_i.

1.2. **The proof of Equation** (1.3) **and of the Kotschick-Morgan conjecture.** The proofs of Equation (1.3) and the Kotschick-Morgan Conjecture [16] both require a partial computation of an intersection number with the link of a closed subspace of a stratified space of the type described in §1.1. For Equation (1.3), the stratified space is $\mathcal{M}_{\mathfrak{t}}/S^1$ and the subspace is $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$. For the Kotschick-Morgan conjecture, the stratified space is the Uhlenbeck compactification of a parameterized moduli space of anti-self-dual connections, $\bar{M}_\kappa^w(g_I)$, parameterized by a path of metrics g_I, while the subspace is $[A_0] \times \mathrm{Sym}^\ell(X)$ where $[A_0]$ is a reducible, anti-self-dual connection.

The strata of the closed subspaces, $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$ and $[A_0] \times \mathrm{Sym}^\ell(X)$, are given by $M_{\mathfrak{s}} \times \Sigma$ and $[A_0] \times \Sigma$, respectively, where $\Sigma \subset \mathrm{Sym}^\ell(X)$ is a smooth stratum.

The gluing theorems of [4] do not quite yield the cone bundle neighborhoods of $M_{\mathfrak{s}} \times \Sigma$ in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ required in Definition 1.2. Rather they provide a *virtual cone bundle neighborhood* by which we mean:

- A cone bundle $\mathrm{Gl}(\mathfrak{t}, \mathfrak{s}, \Sigma) \to M_{\mathfrak{s}} \times \Sigma$,
- An obstruction section, $\mathfrak{o}_\Sigma$, of a pseudo-vector bundle $\Upsilon_{\mathfrak{t},\mathfrak{s}} \to \mathrm{Gl}(\mathfrak{t}, \mathfrak{s}, \Sigma)$,

- A homeomorphism between $\mathfrak{o}_\Sigma^{-1}(0)$ and a neighborhood of $M_\mathfrak{s} \times \Sigma$ in $\bar{\mathcal{M}}_\mathfrak{t}/S^1$.

In [6], we define $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ as the boundary of the union of cone bundles. The actual link $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ is the intersection of $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$ with the zero locus of the obstruction sections. The intersection number in (1.3) is then equal to

$$\langle \mu \smile e(\Upsilon_{\mathfrak{t},\mathfrak{s}}), [\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}] \rangle,$$

where $e(\Upsilon_{\mathfrak{t},\mathfrak{s}})$ acts as an Euler class of the pseudo-vector bundle and μ is a cohomology class dual to the geometric representatives appearing in (1.3). In [6] we show that the virtual cone bundle neighborhoods can be deformed, in a sense to be defined in §4.1, to satisfy the Thom-Mather control condition and the compatible structure group condition of Definition 1.2. Then, the arguments of §1.1 can be applied to compute the cohomology pairing above and thus the intersection number in Equation (1.3).

The methods of [6] also apply to the Kotschick-Morgan conjecture. The gluing maps of Taubes, [29, 3], give the smooth, local cone bundle neighborhoods of $[A_0] \times \Sigma$ in $\bar{M}^w_\kappa(g_I)$. The constructions of [6], while couched in the language of SO(3) monopoles, easily translate to the language of anti-self-dual connections and show that these cone bundle neighborhoods can also be deformed so that they satisfy the Thom-Mather control condition and the compatible structure group condition of Definition 1.2. The pushforward-pullback argument described in §1.1 would then yield a proof of the Kotschick-Morgan conjecture.

While the existence of cone bundle neighborhoods for $[A_0] \times \Sigma$ and the virtual cone bundle neighborhoods for $M_\mathfrak{s} \times \Sigma$ has been known since [29, 3, 4], the result in [6] that these neighborhoods can be deformed to satisfy the Thom-Mather control condition and the compatible structure group condition is new. The authors believe that any attempt to prove Equation (1.3) or the Kotschick-Morgan conjecture must solve these issues.

As an example, in [1], Chen attempts to prove the Kotschick-Morgan conjecture by constructing a bubbletree resolution of the Uhlenbeck compactification $\bar{M}^w_\kappa(g_I)$ and applying equivariant localization arguments to this resolution. To prove that this resolution is a C^1 orbifold, Chen tries to show that, for gluing maps, $\boldsymbol{\gamma}_\Sigma$ and $\boldsymbol{\gamma}_{\Sigma'}$, parameterizing neighborhoods of different strata, the transition map $\boldsymbol{\gamma}_\Sigma^{-1} \circ \boldsymbol{\gamma}_{\Sigma'}$ is smooth. At first glance, this might appear to be a weaker result than the above requirements on Thom-Mather control conditions and compatible structure groups. However, Chen compares the gluing maps $\boldsymbol{\gamma}_\Sigma$ and $\boldsymbol{\gamma}_{\Sigma'}$ by introducing an artificial transition map, $\Phi_{\Sigma,\Sigma'}$, between the domains of the two gluing maps and defining an isotopy between $\boldsymbol{\gamma}_\Sigma$ and $\boldsymbol{\gamma}_{\Sigma'} \circ \Phi_{\Sigma,\Sigma'}$. This artificial transition map, similar to that introduced in [16], has the properties necessary to show Definition 1.2 holds. Thus, a complete implementation of the method of [1] would yield essentially the same program as that of [6], with the additional complication of having to work with the extra data of the bubbletree compactification.

The method of comparing a gluing map $\boldsymbol{\gamma}_\Sigma$ with a composition $\boldsymbol{\gamma}_{\Sigma'}\circ\Phi_{\Sigma,\Sigma'}$ by constructing an isotopy between the two also appears in [20, §4.5.1]. This is a very natural method to try because a direct comparison of the gluing maps $\boldsymbol{\gamma}_\Sigma$ and $\boldsymbol{\gamma}_{\Sigma'}$ appears impractical for reasons discussed in the beginning of §4. The comparison becomes significantly more difficult when there are more than two open sets, for rather than constructing a single isotopy between two maps one must now construct a family of diffeomorphisms parameterized by a higher-dimensional simplex with the maps $\boldsymbol{\gamma}_\Sigma$ and $\boldsymbol{\gamma}_{\Sigma_i}\circ\Phi_{\Sigma,\Sigma_i}$ at the vertices, in a manner similar to the development of a k-isotopy [14, p. 182]. The authors believe that the inductive constructions described in §4.1 could be used to give such a k-isotopy.

1.3. **From Theorem 2.1 to the Witten Conjecture and other applications.** Equation (1.1) shows that the Donaldson invariants are determined by the Seiberg-Witten invariants and the homotopy type of X, but the relation does not immediately yield Witten's formula. We describe how Witten's relation between the Donaldson and Seiberg-Witten invariants follows from Theorem 2.1 in a separate article [7].

As noted in the preceding section, the proof of Equation (1.3) can be adapted to give a proof of the Kotschick-Morgan conjecture.

It is also worth noting that Kronheimer and Mrowka's proof [18] of Property P relies on Equation (1.3).

1.4. **Organization.** This article comprises the following sections. In §2, we review the basic definitions and ideas of the SO(3)-monopole program and describe the intersection numbers in Equation (1.3). In §3, we describe the gluing maps we use to parameterize the neighborhoods containing these intersection numbers. In §4, we summarize the proofs in [6] of the control properties of the gluing map overlaps. Finally, in §5, we use this understanding of the overlaps of the gluing maps to introduce a cohomological formalism to compute the desired intersection numbers.

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2. Preliminaries

Throughout this article, let X be a smooth, closed, and oriented four-manifold. We will assume that $b_1(X)=0$ and $b^+(X)\geq 2$ for the sake of simplicity, although much of what we discuss here holds for more general $b^1(X)$ and for $b^+(X)=1$. The material reviewed in this section appears in full detail in [10, 11, 2].

2.1. **Spin^u structures.** A Clifford module for T^*X is defined by a complex vector bundle $V\to X$ and a Clifford multiplication map, $\rho : T^*X\to \operatorname{End}_{\mathbb{C}}(V)$ which is a real-linear bundle map satisfying

$$\rho(\alpha)^2=-g(\alpha,\alpha)\mathrm{id}_V \quad\text{and}\quad \rho(\alpha)^\dagger=-\rho(\alpha),\quad \alpha\in C^\infty(T^*X). \tag{2.1}$$

The map ρ uniquely extends to a linear isomorphism, $\rho : \Lambda^\bullet(T^*X) \otimes_{\mathbb{R}} \mathbb{C} \to \mathrm{End}_{\mathbb{C}}(V)$, and gives each fiber V_x the structure of a Hermitian Clifford module for the complex Clifford algebra $\mathbb{C}\ell(T^*X)|_x$, for all $x \in X$. There is a splitting $V = V^+ \oplus V^-$, where $V^\pm$ are the ∓ 1 eigenspaces of $\rho(\mathrm{vol})$.

A *spin*^{c} *structure* is then a Clifford module for T^*X, $\mathfrak{s} = (\rho, W)$, where W has complex rank four; it defines a class

$$c_1(\mathfrak{s}) = c_1(W^+) \in H^2(X; \mathbb{Z}). \tag{2.2}$$

We call a Clifford module for T^*X, consisting of a pair $\mathfrak{t} \equiv (\rho, V)$, a *spin*^{u} *structure* when V has complex rank eight. If $\mathfrak{s} = (\rho_W, W)$ is a spin^c structure, then for any spin^u structure, $\mathfrak{t} = (\rho, V)$, there is a complex, rank-two vector bundle $E \to X$ with $(\rho, V) = (\rho_W \otimes \mathrm{id}_E, W \otimes E)$ [10, Lemma 2.3].

A spin^u structure, $\mathfrak{t} = (\rho, V)$, defines some auxiliary bundles over X. Recall that $\mathfrak{g}_V \subset \mathfrak{su}(V)$ is the SO(3) subbundle given by the span of the sections of the bundle $\mathfrak{su}(V)$ which commute with the action of $\mathbb{C}\ell(T^*X)$ on V. The fibers V_x^+ define complex lines whose tensor-product square is $\det(V_x^+)$ and thus a complex line bundle over X,

$$\det{}^{\frac{1}{2}}(V^+). \tag{2.3}$$

A spin^u structure $\mathfrak{t}$ thus defines classes,

$$c_1(\mathfrak{t}) = \tfrac{1}{2} c_1(V^+), \quad p_1(\mathfrak{t}) = p_1(\mathfrak{g}_V), \quad \text{and} \quad w_2(\mathfrak{t}) = w_2(\mathfrak{g}_V). \tag{2.4}$$

If $\mathfrak{s} = (\rho, W)$ is a spin^c structure and $V \cong W \otimes_{\mathbb{C}} E$, then

$$\mathfrak{g}_V = \mathfrak{su}(E) \quad \text{and} \quad \det{}^{\frac{1}{2}}(V^+) = \det(W^+) \otimes_{\mathbb{C}} \det(E).$$

2.2. **The equations.** A unitary connection A on V is *spin* if

$$[\nabla_A, \rho(\alpha)] = \rho(\nabla \alpha) \quad \text{on } C^\infty(V), \tag{2.5}$$

for any $\alpha \in C^\infty(T^*X)$, where ∇ is the Levi-Civita connection. We will write $\mathcal{A}_{\mathfrak{t}}$ for the space of spin connections on V which induce a fixed connection on $\det(V)$. There is a bijection between $\mathcal{A}_{\mathfrak{t}}$ and the space of orthogonal connections on $\mathfrak{g}_V$.

We will write $\mathcal{G}_{\mathfrak{t}}$ for the space of gauge transformations of V which commute with Clifford multiplication and which have Clifford-determinant equal to one [10, Definition 2.6]. If $V = W \otimes E$ as above, then $\mathcal{G}_{\mathfrak{t}}$ is the set of gauge transformations of V induced by the special-unitary gauge transformations of E.

We will consider pairs in the Sobolev completions of the space

$$\tilde{\mathcal{C}}_{\mathfrak{t}} = \mathcal{A}_{\mathfrak{t}} \times \Omega^0(V^+),$$

and points in the quotient,

$$\mathcal{C}_{\mathfrak{t}} = \tilde{\mathcal{C}}_{\mathfrak{t}} / \mathcal{G}_{\mathfrak{t}}.$$

For $(A, \Phi) \in \tilde{\mathcal{C}}_{\mathfrak{t}}$, the SO(3)-monopole equations are then:

$$\begin{aligned} \rho(F_A^+)_0 &= (\Phi \otimes \Phi^*)_{00}, \\ D_A \Phi &= 0. \end{aligned} \tag{2.6}$$

Here, $D_A : \Omega^0(V^+) \to \Omega^0(V^-)$ is the Dirac operator associated to the connection A, while $(\Phi \otimes \Phi^*)_{00}$ denotes the doubly trace-free component of $\Phi \otimes \Phi^*$ (see [10] for details). The moduli space of SO(3) monopoles on $\mathfrak{t}$ is

$$\mathcal{M}_{\mathfrak{t}} = \{[A, \Phi] \in \mathcal{C}_{\mathfrak{t}} : (2.6) \text{ holds}\}.$$

We write $\mathcal{M}_{\mathfrak{t}}^0$ for the subspace of $\mathcal{M}_{\mathfrak{t}}$ where the section Φ is not identically zero. We let $\mathcal{M}_{\mathfrak{t}}^*$ denote the set of pairs $[A, \Phi] \in \mathcal{M}_{\mathfrak{t}}$ where A does not admit a reduction as $A = A_1 \oplus A_2$ with respect to a splitting $V = (W \otimes L_1) \oplus (W \otimes L_2)$ for line bundles L_1 and L_2. Then, for generic perturbations [2, 30], the space

$$\mathcal{M}_{\mathfrak{t}}^{*,0} = \mathcal{M}_{\mathfrak{t}}^* \cap \mathcal{M}_{\mathfrak{t}}^0$$

is a smooth manifold.

2.3. **The singularities.** The S^1 action,

$$\left(e^{i\theta}, [A, \Phi]\right) \mapsto [A, e^{i\theta}\Phi] \tag{2.7}$$

has stabilizer $\{\pm 1\}$ on $\mathcal{M}_{\mathfrak{t}}^{*,0}$. There are two types of fixed points for this action. To describe these, it helps to assume that $V = W \otimes E$ where $\mathfrak{s} = (\rho, W)$ is a spin^c structure.

The first type of fixed point occurs when the section Φ is identically zero. The subspace of such pairs is diffeomorphic to the moduli space of anti-self-dual connections on the SO(3) bundle $\mathfrak{g}_V = \mathfrak{su}(E)$:

$$\mathcal{M}_{\mathfrak{t}} - \mathcal{M}_{\mathfrak{t}}^0 = M_\kappa^w,$$

where M_κ^w is the moduli space of anti-self-dual connections on $\mathfrak{g}_V$, $w = c_1(E)$, and $\kappa = p_1(\mathfrak{t})$ [10].

The second type of fixed point occurs when the connection A is reducible in the sense that it can be written as $A = A_1 \oplus A_2$ with respect to a splitting

$$V \simeq W \otimes E \simeq (W \otimes L_1) \oplus (W \otimes L_2).$$

The subspace of such fixed points is diffeomorphic to a perturbation of the Seiberg-Witten moduli space associated to the spin^c structure $\mathfrak{s} \otimes L_i = (\rho, W \otimes L_i)$ (either $i = 1$ or $i = 2$) as discussed in [10]. These spin^c structures lie in the set

$$\operatorname{Red}(\mathfrak{t}) = \{\mathfrak{s} \in \operatorname{Spin}^c(X) : (c_1(\mathfrak{s}) - c_1(\mathfrak{t}))^2 = p_1(\mathfrak{t})\}.$$

If $M_{\mathfrak{s}}$ is the Seiberg-Witten moduli space associated to the spin^c structure $\mathfrak{s}$, then

$$\mathcal{M}_{\mathfrak{t}} - \mathcal{M}_{\mathfrak{t}}^* = \cup_{\mathfrak{s} \in \operatorname{Red}(\mathfrak{t})} M_{\mathfrak{s}}.$$

The moduli space $\mathcal{M}_{\mathfrak{t}}/S^1$ then gives a cobordism between the links, $\mathbf{L}_{\mathfrak{t}}^{\mathrm{asd}}$ of M_κ^w in $\mathcal{M}_{\mathfrak{t}}$ and $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ of $M_{\mathfrak{s}}$ in $\mathcal{M}_{\mathfrak{t}}$. The dimension of this cobordism is,

$$\dim \mathcal{M}_{\mathfrak{t}}^{*,0}/S^1 = \dim M_\kappa^w + 2n_a(\mathfrak{t}) - 2, \tag{2.8}$$

where

$$n_a(\mathfrak{t}) = \mathrm{Index}_{\mathbb{C}}\, D_A = \frac{1}{4}\left(p_1(\mathfrak{t}) + c_1(\mathfrak{t})^2 - \sigma(X)\right). \tag{2.9}$$

Unfortunately, the cobordism $\mathcal{M}_{\mathfrak{t}}/S^1$ is not compact.

2.4. **The compactification.** The moduli space $\mathcal{M}_{\mathfrak{t}}$ admits a compactification similar to the Uhlenbeck compactification of the moduli space of anti-self-dual connections. Let $\mathfrak{t}(\ell)$ be the spin^u structure with $c_1(\mathfrak{t}(\ell)) = c_1(\mathfrak{t})$ and $p_1(\mathfrak{t}(\ell)) = p_1(\mathfrak{t})+4\ell$. Thus, if we can write $V = W\otimes E$, where $\mathfrak{s} = (\rho, W)$ is a spin^c structure, then $\mathfrak{t}(\ell) = (\rho, W\otimes E_\ell)$ where $c_1(E_\ell) = c_1(E)$ and $c_2(E_\ell) = c_2(E) - \ell$. Let $\mathrm{Sym}^\ell(X)$ denote the ℓ-th symmetric product of X, $\mathrm{Sym}^\ell(X) = X^\ell/\mathfrak{S}_\ell$. Define

$$I\mathcal{M}_{\mathfrak{t}} = \mathcal{M}_{\mathfrak{t}} \cup \left(\cup_{\ell=1}^N \mathcal{M}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^\ell(X)\right),$$

and give this space the topology induced by Uhlenbeck convergence of sequences. That is, a sequence $\{[A_\alpha, \Phi_\alpha]\}$ converges to a point $([A_0,\Phi_0],\mathbf{x})$ where $\mathbf{x} \in \mathrm{Sym}^\ell(X)$ if, after gauge transformations, the restrictions of (A_α, Φ_α) to compact subsets of $X - \mathbf{x}$ converge to the same restriction of (A_0,Φ_0) in the smooth topology. In addition, the sequence of measures defined by $|F_{A_\alpha}|^2$ must converge, in the weak star topology, to that defined by $|F_{A_0}|^2$ added to a multiple of the Dirac delta measure supported at $\mathbf{x}$.

Define $\bar{\mathcal{M}}_{\mathfrak{t}}$ to be the closure of $\mathcal{M}_{\mathfrak{t}}$ in $I\mathcal{M}_{\mathfrak{t}}$ with the topology described above.

The dimension formula in Equation (2.8) implies that

$$\dim\left(\mathcal{M}_{\mathfrak{t}(\ell)}^{*,0} \times \mathrm{Sym}^\ell(X)\right) = \dim \mathcal{M}_{\mathfrak{t}}^{*,0} - 2\ell, \tag{2.10}$$

which will be useful in dimension-counting arguments.

We will also use the space

$$\bar{\mathcal{C}}_{\mathfrak{t}} = \mathcal{C}_{\mathfrak{t}} \cup \left(\cup_{\ell=1}^N \mathcal{C}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^\ell(X)\right), \tag{2.11}$$

with the same definition of convergence.

The S^1 action (2.7) extends continuously to the compactification $\bar{\mathcal{M}}_{\mathfrak{t}}$ with similar fixed point sets. That is, the fixed point subspaces again divide into two types, those where the section vanishes and those where the connection is reducible. The subspace of $\bar{\mathcal{M}}_{\mathfrak{t}}$ where the section vanishes is given by $\bar{M}_\kappa^w$, the Uhlenbeck compactification of the moduli space of anti-self-dual connections on $\mathfrak{g}_V$.

The fixed point subspaces where the connection is reducible are more complicated when they lie in the compactification. A level-ℓ reducible is of the form

$$M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X) \subset \mathcal{M}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^\ell(X),$$

where $M_{\mathfrak{s}} \subset \mathcal{M}_{\mathfrak{t}(\ell)}$ and thus $(c_1(\mathfrak{s}) - c_1(\mathfrak{t}))^2 = p_1(t) + 4\ell$. We then define

$$\begin{aligned}\operatorname{Red}_\ell(\mathfrak{t}) &= \{\mathfrak{s} \in \operatorname{Spin}^c(X) : (c_1(\mathfrak{s}) - c_1(\mathfrak{t}))^2 = p_1(\mathfrak{t}) + 4\ell\},\\ \bar{\operatorname{Red}}(\mathfrak{t}) &= \cup_{\ell \geq 0} \operatorname{Red}_\ell(\mathfrak{t}).\end{aligned}$$

The space $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ defines a smoothly-stratified cobordism between the link of $\bar{M}^w_\kappa$ in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ and the links of $M_{\mathfrak{s}} \times \operatorname{Sym}^\ell(X)$ in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$, where $\mathfrak{s} \in \bar{\operatorname{Red}}(\mathfrak{t})$.

2.5. **The cobordism and cohomology classes.** Let $\bar{\mathbf{L}}^{\mathrm{asd}}_{\mathfrak{t}}$ be the link of the Uhlenbeck compactification of the moduli space of anti-self-dual connections, $\bar{M}^w_\kappa$, in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$. For $\mathfrak{s} \in \bar{\operatorname{Red}}_\ell(\mathfrak{t})$, let $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ be the link of $M_{\mathfrak{s}} \times \operatorname{Sym}^\ell(X)$ in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$.

Following [17] and recalling that we assumed $b_1(X) = 0$, define

$$\mathbb{A}(X) = \operatorname{Sym}^\bullet(H_2(X;\mathbb{R}) \oplus H_0(X;\mathbb{R})),$$

and assign to $\beta \in H_\bullet(X)$ the degree $\deg(\beta) = 4 - \bullet$ in $\mathbb{A}(X)$. For a monomial z in $\mathbb{A}(X)$, a geometric representative $\bar{\mathcal{V}}(z)$ was constructed in [17] whose intersection number with $\bar{M}^w_\kappa$ defined the Donaldson invariant. In [10, 11], geometric representatives extending $\bar{\mathcal{V}}(z)$ from $\bar{M}^w_\kappa$ to $\bar{\mathcal{M}}^*_{\mathfrak{t}}/S^1$ were defined. In addition, a geometric representative $\bar{\mathcal{W}}$ was defined on $\bar{\mathcal{M}}^{*,0}_{\mathfrak{t}}/S^1$ which is dual to a multiple of the first Chern class of the S^1 action.

For $\deg(z) = \dim M^w_\kappa$, we computed the following intersection number in [11],

$$2^{n-1} D^w_X(z) = \#\left(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{n-1} \cap \bar{\mathbf{L}}^{\mathrm{asd}}_{\mathfrak{t}}\right),$$

where $n = n_a(\mathfrak{t})$. The geometric representatives intersect the lower strata of $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ transversely away from the fixed point sets of the S^1 action. Hence, the dimension formula (2.10) shows that the cobordism provided by $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ and the preceding identity yield the following expression for the Donaldson invariant:

$$2^{n-1} D^w_X(z) = -\sum_{\mathfrak{s} \in \operatorname{Spin}^c(X)} \#\left(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{n-1} \cap \mathbf{L}_{\mathfrak{t},\mathfrak{s}}\right), \tag{2.12}$$

where we define the link $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ to be empty if $\mathfrak{s} \notin \bar{\operatorname{Red}}(\mathfrak{t})$. The computation of the intersection number

$$\#\left(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{n-1} \cap \mathbf{L}_{\mathfrak{t},\mathfrak{s}}\right) \tag{2.13}$$

appears in [11] for $\mathfrak{s} \in \operatorname{Red}(\mathfrak{t})$ and appears in [12] for $\mathfrak{s} \in \operatorname{Red}_1(\mathfrak{t})$. For $\mathfrak{s} \in \operatorname{Red}_\ell(\mathfrak{t})$ with $\ell \geq 2$, the computation becomes significantly more difficult, although the development in [20] suggests an approach to the case $\ell = 2$. In [6], we show how the following version of Equation (1.3) follows from the technical result Theorem 4.2.

Theorem 2.1. *Assume the result of Theorem 4.2. If* $b_1(X) = 0$, *if* $\mathfrak{t}$ *is a spin*^{u} *structure on* X *and* $\mathfrak{s} \in$ *Spin*$^c(X)$ *with*

$$M_{\mathfrak{s}} \times \operatorname{Sym}^\ell(X) \subset I\mathcal{M}_{\mathfrak{t}},$$

then

$$\begin{aligned}\#&\left(\bar{\mathcal{V}}(h^{\delta-2m}x^m)\cap\bar{\mathcal{W}}^{n-1}\cap\mathbf{L}_{\mathfrak{t},\mathfrak{s}}\right)\\&=SW_X(\mathfrak{s})\sum_{i=0}^{k}p_{\delta,\ell,m,i}(A,B)Q_X^i(h),\end{aligned}$$

where $k=\min[\ell,(\delta-2m)/2)]$,

$$\begin{aligned}A&=\langle c_1(\mathfrak{s})-c_1(\mathfrak{t}),h\rangle,\\B&=\langle c_1(\mathfrak{t}),h\rangle,\end{aligned}$$

and $p_{\delta,\ell,m,i}$ *is a homogeneous polynomial of degree* $(\delta-2m-2i)$ *whose coefficients are universal functions of*

$$\chi(X),\ \sigma(X),\ c_1(\mathfrak{s})^2,\ c_1(\mathfrak{t})^2,\ c_1(\mathfrak{t})\cdot c_1(\mathfrak{s}),\ p_1(\mathfrak{t}),\ m,\ \delta,\ \textit{and}\ \ell.$$

Remark 2.2. A similar result holds when $b^1(X)>0$, but we omit it for simplicity.

3. The gluing maps

To prove Theorem 2.1, we show that for each smooth stratum Σ of $\operatorname{Sym}^\ell(X)$, the subspaces $M_{\mathfrak{s}}\times\Sigma$ of $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ admit virtual local cone bundle neighborhoods in the sense of §1.2 which satisfy Definition 1.2. The gluing theorems of [4] provide these neighborhoods by defining a "gluing map" which parameterizes a neighborhood of $M_{\mathfrak{s}}\times\Sigma$ in $\bar{\mathcal{M}}_{\mathfrak{t}}$.

A gluing map is a composition of a splicing map and a gluing perturbation which we describe in the following sections.

3.1. **Splicing maps.**

The splicing map creates an approximate solution to the SO(3)-monopole equations by attaching solutions in $\mathcal{M}_{\mathfrak{t}(\ell)}$ to concentrated solutions on S^4 in a kind of connected sum construction. The domain of the splicing map contains the following data:

1. A 'background pair' $(A_0,\Phi_0)\in\mathcal{M}_{\mathfrak{t}(\ell)}$,
2. Splicing points, $\mathbf{x}\in\Sigma$,
3. Solutions of the equations on S^4.

We now describe the spaces in which these data live.

3.1.1. *The background pair.* In [10], a neighborhood of $M_{\mathfrak{s}}$ in $\mathcal{M}_{\mathfrak{t}(\ell)}$ was described using virtual neighborhood techniques. That is, we defined

1. A pair of vector bundles, $N_{\mathfrak{t}(\ell),\mathfrak{s}}\to M_{\mathfrak{s}}$ and $\Upsilon_{\mathfrak{s}}\to M_{\mathfrak{s}}$,
2. An S^1-equivariant embedding $\gamma_{\mathfrak{s}}:N_{\mathfrak{t}(\ell),\mathfrak{s}}\to\mathcal{C}_{\mathfrak{t}(\ell)}$ which is the identity on $M_{\mathfrak{s}}$,
3. A section $\mathfrak{o}_{\mathfrak{s}}$ of the pullback of $\Upsilon_{\mathfrak{s}}$ to $N_{\mathfrak{t}(\ell),\mathfrak{s}}$

such that the restriction of $\boldsymbol{\gamma}_{\mathfrak{s}}$ to $\mathfrak{o}_{\mathfrak{s}}^{-1}(0)$ defines an S^1-equivariant homeomorphism from $\mathfrak{o}_{\mathfrak{s}}^{-1}(0)$ onto a neighborhood of $M_{\mathfrak{s}}$ in $\mathcal{M}_{\mathfrak{t}(\ell)}$.

3.1.2. *The splicing points.* Let $\Sigma \subset \mathrm{Sym}^\ell(X)$ be a stratum given by the partition $\kappa_1 + \cdots + \kappa_r = \ell$. A point $\mathbf{x} \in \Sigma$ can be described by an unordered collection of distinct points $\{x_1, \ldots, x_r\} \subset X$ with the multiplicity κ_i attached to x_i. We will write $\Sigma < \Sigma'$ to indicate that $\Sigma \subset \mathrm{cl}(\Sigma')$.

3.1.3. *The S^4 connections.* Solutions of the SO(3)-monopole equations on S^4 correspond to anti-self-dual connections on an SU(2) bundle $E_\kappa \to S^4$. In addition, the splicing process requires a frame for $E_\kappa|_s$ where $s \in S^4$ is the south pole. We write $M^s_\kappa(S^4)$ for the moduli space of framed, anti-self-dual connections on E_κ.

We also require these connections to be mass-centered, in the sense that

$$\int_{S^4} \vec{x}|F_A|^2 d\,\mathrm{vol} = 0,$$

where $\vec{x} : S^4 - \{s\} \to \mathbb{R}^4$ is given by stereographic projection.

Finally, we require these connections to be concentrated near the north pole in the sense that

$$\int_{S^4} |\vec{x}|^2 |F_A|^2 d\,\mathrm{vol} \leq \varepsilon, \tag{3.1}$$

for some small constant ε which shall not be specified here.

The connections on S^4 then lie in the space,

$$\bar{M}^{s,\diamond}_\kappa(S^4),$$

which is defined to be the Uhlenbeck compactification of the moduli space of framed, mass-centered, sufficiently concentrated, anti-self-dual connections on $E_\kappa \to S^4$. The space $\bar{M}^{s,\diamond}_\kappa(S^4)$ is a cone with cone parameter squared given by the left-hand side of the inequality (3.1). In addition, one can show that $\bar{M}^{s,\diamond}_\kappa(S^4)$ is a sub-analytic and thus a Thom-Mather stratified space.

3.1.4. *Describing the map.* Let $\Sigma \subset \mathrm{Sym}^\ell(X)$ be the stratum given by the partition $\ell = \kappa_1 + \cdots + \kappa_r$. Let $\mathbf{A}$ denote the vector of data prescribed by

(1) $(A_0, \Phi_0) \in \boldsymbol{\gamma}_{\mathfrak{s}}(N_{\mathfrak{t}(\ell),\mathfrak{s}})$,
(2) $\mathbf{x} \in \Sigma$, with $\mathbf{x} = \{x_1, \ldots, x_r\} \subset X$, and
(3) For $i = 1, \ldots, r$, $[A_i, F^s_i] \in \bar{M}^{s,\diamond}_{\kappa_i}(S^4)$.

We define the image of the splicing map to be

$$(A', \Phi') = \boldsymbol{\gamma}'_\Sigma(\mathbf{A}) = (A_0, \Phi_0)\#_{x_1}(A_1, 0)\# \ldots \#_{x_r}(A_r, 0),$$

where, roughly speaking,

$$(A', \Phi')(x) \approx \begin{cases} (A_0, \Phi_0)(x) & \text{for } x \text{ away from } x_i, \\ (A_i, 0)(x) & \text{for } x \text{ near } x_i. \end{cases}$$

The precise definition of the splicing map requires careful use of trivializations of the relevant bundles and cut-off functions to interpolate between the given pairs of sections and connections. Nonetheless, it is a completely explicit map.

3.1.5. *The domain of the map.* If the stratum Σ is given by the partition $\kappa_1 + \cdots + \kappa_r = \ell$, then the domain, $\mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma)$, of the splicing map $\boldsymbol{\gamma}'_\Sigma$ described in §3.1.4 can be described as a fiber bundle:

$$\begin{array}{ccc}\prod_{i=1}^r \bar{M}^{s,\diamond}_{\kappa_i}(S^4) & \longrightarrow & \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma) \\ & & \Big\downarrow{\scriptstyle \Pi_\Sigma} \\ & & N_{\mathfrak{t}(\ell),\mathfrak{s}} \times \Sigma \end{array} \tag{3.2}$$

with underlying principle bundle determined by the homotopy type of X and the characteristic classes $c_1(\mathfrak{s})$, $c_1(\mathfrak{t})$, $p_1(\mathfrak{t})$. Note that because the spaces $\bar{M}^{s,\diamond}_{\kappa_i}(S^4)$ are cones, the fiber bundle (3.2) is a cone bundle in the sense of Definition 1.2. We further note that (3.2) is a non-trivial bundle because of the need to chose trivializations of the relevant bundles in the splicing map. We will write π_Σ for the composition

$$\pi_\Sigma : \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma) \to N_{\mathfrak{t}(\ell),\mathfrak{s}} \times \Sigma \to M_\mathfrak{s} \times \Sigma \tag{3.3}$$

of the projection in (3.2) with the obvious projection defined by the vector bundle $N_{\mathfrak{t}(\ell),\mathfrak{s}} \to M_\mathfrak{s}$.

3.2. **Gluing perturbations.** The pair $\boldsymbol{\gamma}'_\Sigma(\mathbf{A})$ described in the preceding section is not a solution of the SO(3)-monopole equations, but it is almost a solution. That is, if we write the SO(3)-monopole equations as a map to a suitable Banach space,

$$\mathfrak{S} : \tilde{\mathcal{C}}_\mathfrak{t} \to \mathbf{B},$$

then the norm $\|\mathfrak{S}(\boldsymbol{\gamma}'_\Sigma(\mathbf{A}))\|$ is small. It is possible to deform the image of the splicing map so that the deformed image contains a neighborhood of $M_\mathfrak{s} \times \Sigma$ in $\bar{\mathcal{M}}_\mathfrak{t}$ as follows.

First, the linearization of the map $\mathfrak{S}$ is not surjective. The cokernel of the linearization can be stabilized with a vector bundle,

$$\Upsilon_\Sigma \to \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma),$$

in the sense that the fiber of Υ_Σ over $(A',\Phi') \in \mathrm{Im}(\boldsymbol{\gamma}'_\Sigma)$ contains the cokernel of the linearization of $\mathfrak{S}$ at (A',Φ'). There is then, up to gauge transformation, a unique solution, $\mathfrak{p}_\Sigma(A',\Phi')$, of the equation

$$\mathfrak{S}((A',\Phi') + \mathfrak{p}_\Sigma(A',\Phi')) \in \Upsilon_\Sigma|_{(A',\Phi')},$$

We call $\mathfrak{p}(A',\Phi')$ the *gluing perturbation* and the map $(A',\Phi') \mapsto \mathfrak{p}(A',\Phi')$ is smooth. The gluing map is then defined by

$$\boldsymbol{\gamma}_\Sigma = \boldsymbol{\gamma}'_\Sigma + \mathfrak{p} \circ \boldsymbol{\gamma}'_\Sigma.$$

The *obstruction section*, $\mathfrak{o}_\Sigma$, of Υ_Σ is defined by $\mathfrak{S} \circ \boldsymbol{\gamma}_\Sigma$. The construction of the map $\mathfrak{p}$ appears in [4] and it follows that

Theorem 3.1. *The restriction of the gluing map $\boldsymbol{\gamma}_\Sigma$ to the zero-locus of the obstruction map, $\mathfrak{o}_\Sigma^{-1}(0)$, parameterizes a neighborhood of $M_\mathfrak{s} \times \Sigma$ in $\bar{\mathcal{M}}_\mathfrak{t}$.*

4. Understanding the overlaps

When $\ell = 1$, Theorem 3.1 is all we need to compute the intersection number (2.13) because there is only one stratum $X = \mathrm{Sym}^1(X)$. In [12], we argue that this intersection number can be written as

$$\#\left(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{n-1} \cap \mathbf{L}_{\mathfrak{t},\mathfrak{s}}\right) = \langle e(\Upsilon_\Sigma) \smile \bar{\mu}_p(z) \smile \bar{\mu}_c^{n-1}, [\partial\, \mathrm{Gl}(\mathfrak{t},\mathfrak{s},X)]\rangle,$$

where $\mathrm{Gl}(\mathfrak{t},\mathfrak{s},X)$ is defined in (3.2). We use our understanding of the structure group of the fibration (3.2) and a pushforward formula for the map π_X defined in (3.3) to compute the above cohomological pairing.

For $\ell > 1$, we cannot use Theorem 3.1 and the information on the local bundle (3.2) given in §3.1 to compute the intersection number (2.13) because more than one gluing map is necessary to cover a neighborhood of $M_{\mathfrak{s}} \times \mathrm{Sym}^\ell(X)$. One needs to understand the overlap of these maps to do a cohomological computation. That is, one cannot just add up the contributions from each open set as there might be intersection points in the overlap of two or more maps. Such problems, with only two maps needed, have been addressed in [25, 19, 20]. This problem becomes significantly more difficult with three or more gluing maps are involved.

The difficulty in understanding these overlaps arises largely from the definition of the gluing perturbation $\mathfrak{p}$. That is, while the splicing map $\boldsymbol{\gamma}'_\Sigma$ is quite explicit, the gluing perturbation $\mathfrak{p}$ arises from an implicit function theorem argument and is thus not sufficiently explicit for us to compute $\boldsymbol{\gamma}_\Sigma^{-1} \circ \gamma_{\Sigma'}$.

The idea underlying [6] is to show that images of the *splicing maps*, rather than the images of the gluing maps, satisfy the conditions of Definition 1.2. As the splicing maps are presently defined, this approach might seem problematic because the images of the splicing maps of two different strata need not intersect at all, even when the images of the associated gluing maps do. In [6], we introduce deformations of the domains of the splicing maps and of the maps themselves to get splicing maps, $\boldsymbol{\gamma}''_\Sigma$,whose overlaps are controlled by push-out diagrams of the form

$$\begin{array}{ccc} \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma,\Sigma') & \xrightarrow{\rho^d_{\Sigma,\Sigma'}} & \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma) \\ {\scriptstyle \rho^u_{\Sigma,\Sigma'}}\big\downarrow & & {\scriptstyle \gamma''_\Sigma}\big\downarrow \\ \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma') & \xrightarrow{\gamma''_{\Sigma'}} & \bar{\mathcal{C}}_{\mathfrak{t}} \end{array} \tag{4.1}$$

where the maps $\rho^d_{\Sigma,\Sigma'}$ and $\rho^u_{\Sigma,\Sigma'}$ are open embeddings and $\bar{\mathcal{C}}_{\mathfrak{t}}$ is defined in (2.11). We describe the deformation of the domain in §4.1 and we describe the deformation of the splicing map in §4.2.

4.1. **Deforming the fiber.**

The deformation of the domain of the splicing map mentioned above is a deformation of the fiber of the diagram (3.2). We construct the *spliced ends moduli space* $\bar{M}^{s,\diamond}_{spl,\kappa}(S^4)$ as a deformation of the moduli space $\bar{M}^{s,\diamond}_{\kappa}(S^4)$ described in §3.1.3. The deformation consists

of replacing neighborhoods of the punctured trivial strata with images of splicing maps. By punctured trivial strata, we mean the subspace,

$$\{[\Theta]\} \times \left(\mathrm{Sym}^{\kappa,\diamond}(\mathbb{R}^4) - \{c_\kappa\}\right) \subset \bar{M}^{s,\diamond}_\kappa(S^4), \tag{4.2}$$

where Θ is the trivial connection, $\mathrm{Sym}^{\kappa,\diamond}(\mathbb{R}^4)$ is the subspace of the symmetric product given by points with center-of-mass equal to zero, and $c_\kappa \in \mathrm{Sym}^{\kappa,\diamond}(\mathbb{R}^4)$ is the point supported entirely at the origin in $\mathbb{R}^4$.

A neighborhood of the stratum $\{[\Theta]\} \times \Sigma$ in $\bar{M}^{s,\diamond}_\kappa(S^4)$ is parameterized by a gluing map,

$$\boldsymbol{\gamma}_{S^4,\Sigma} : \Sigma \times \prod_{i=1}^r \bar{M}^{s,\diamond}_{\kappa_i}(S^4) \to \bar{M}^{s}_\kappa(S^4), \tag{4.3}$$

where $\kappa_i + \cdots + \kappa_r = \kappa$ is the partition of κ determining the stratum κ. (Note that the domain of $\boldsymbol{\gamma}_{S^4,\Sigma}$ is actually twisted by a symmetric group action, but that is not relevant to this discussion.) We will write

$$\mathcal{N}_\kappa \subset \bar{M}^{s,\diamond}_{\kappa_i}(S^4)$$

for the neighborhood of the subspace (4.2) given by the union of the images of the gluing maps $\boldsymbol{\gamma}_{S^4,\Sigma}$.

Each gluing map $\boldsymbol{\gamma}_{S^4,\Sigma}$ is a deformation of a splicing map $\boldsymbol{\gamma}'_{S^4,\Sigma}$ defined as in §3.1. Hence, there is an isotopy $\Gamma_{S^4,\Sigma}$ with $\Gamma_{S^4,\Sigma}(1,\cdot) = \boldsymbol{\gamma}_{S^4,\Sigma}(\cdot)$ and $\Gamma_{S^4,\Sigma}(0,\cdot) = \boldsymbol{\gamma}'_{S^4,\Sigma}(\cdot)$. Make the obvious extension of $\Gamma_{S^4,\Sigma}(t,\cdot)$ to all $t \in \mathbb{R}$. We wish to write

$$W_\kappa = \cup_\Sigma \mathrm{Im}(\boldsymbol{\gamma}'_{S^4,\Sigma}), \tag{4.4}$$

and then to define a collar of ∂W_κ by a function $\lambda : W_\kappa \to [0,1]$ such that $\lambda^{-1}([0,1/2))$ contains (4.2) and $\lambda^{-1}(1) = \partial W_\kappa$. Then, we wish to replace $\mathcal{N}_\kappa$ with a deformation of W_κ, defined by replacing a point $\boldsymbol{\gamma}_{S^4,\Sigma}(\mathbf{A})$ with $\Gamma_{S^4,\Sigma}(2\lambda(\mathbf{A}) - 1, \mathbf{A})$.

The problem with the above description is that the space W_κ is not homeomorphic to $\mathcal{N}_\kappa$ because the images of the splicing maps defined by different strata Σ need not intersect. The deformation of W_κ thus need not be an isotopy and the resulting space need not be smoothly-stratified.

We repair this argument inductively. We begin by observing that the splicing map has an associative property which allows us, assuming the connections in the domain of $\boldsymbol{\gamma}'_{S^4,\Sigma}$ are themselves the image of a splicing map, to control the overlaps of different splicing maps. This control then makes W_κ a smoothly stratified space homeomorphic to $\mathcal{N}_\kappa$, competing the naive argument above.

4.1.1. *Associativity of the splicing map.* Splicing connections on S^4 to the trivial connection on S^4 has a nice property which we refer to as the *associativity of splicing*. It can be summarized as follows. Let $A_{i,j}$ be connections on bundles $E_{\kappa_{i,j}} \to S^4$. Let $x_{i,j}$ and y_i be distinct points in $\mathbb{R}^4$. Define

$$A_i = \Theta \#_{x_{i,1}} A_{i,1} \# \dots \#_{x_{i,r_i}} A_{i,s_i}$$

to be the connection obtained by splicing the connections $A_{i,j}$ to the trivial connection Θ at the points $x_{i,j}$. Then, if the connections $A_{i,j}$ are sufficiently concentrated near the north pole in the sense of (3.1) relative to the separation of the points $x_{i,j}$, one has

$$\begin{aligned}&\Theta\#_{y_1}A_1\#\dots\#_{y_r}A_r\\ &\quad=\Theta\#_{y_1+x_{1,1}}A_{1,1}\#\dots\#_{y_i+x_{i,j}}A_{i,j}\#\dots\#_{y_r+x_{r,s_r}}A_{r,s_r}\end{aligned}\tag{4.5}$$

Equation (4.5) identifies the composition of two splicing maps with a single splicing map. This identity is crucial in the construction described in the following section. We note that it is our ability to write down the splicing map explicitly that makes it possible to obtain Equation (4.5).

4.1.2. *Overlapping splicing maps.* We now define the spliced-end moduli space, $\bar{M}^{s,\diamond}_{spl,\kappa}(S^4)$. This moduli space is a deformation of $\bar{M}^{s,\diamond}_{\kappa}(S^4)$ with the property that a neighborhood of the subspace (4.2) is given by the image of splicing maps instead of the gluing maps in (4.3). This construction uses induction on κ. For $\kappa=1$, the trivial strata (4.2) is empty because of the absence of the cone point c_1 from (4.2), and we may define

$$\bar{M}^{s,\diamond}_{spl,1}(S^4)=\bar{M}^{s,\diamond}_{1}(S^4).$$

Because the cone point, c_κ, is not included in the strata in (4.2), the domains of the gluing maps (4.3) contain only moduli spaces $\bar{M}^{s,\diamond}_{\kappa_i}(S^4)$ with $\kappa_i<\kappa$. Thus, using induction we may require the neighborhood of the punctured trivial strata to be parameterized by the splicing map with domain

$$\Sigma\times\prod_{i=1}^{r}\bar{M}^{s,\diamond}_{spl,\kappa_i}(S^4),\tag{4.6}$$

instead of the domain of the gluing map (4.3). The associativity of splicing to the trivial connection (4.3), then ensures that the overlap of the images of two such splicing maps can be understood by the following pushout diagram:

$$\begin{tikzcd}\nu(\Sigma,\Sigma')\times\prod_{i,j}\bar{M}^{s,\diamond}_{spl,\kappa_{i,j}}(S^4) \arrow[r, "\rho^u_{\Sigma,\Sigma'}(S^4)"] \arrow[d, "\rho^d_{\Sigma,\Sigma'}(S^4)"'] & \Sigma'\times\prod_{i,j}\bar{M}^{s,\diamond}_{spl,\kappa_{i,j}}(S^4) \arrow[d, "\gamma_{\Sigma'}(S^4)'"'] \\ \Sigma\times\prod_{i}\bar{M}^{s,\diamond}_{spl,\kappa_i}(S^4) \arrow[r, "\gamma_\Sigma(S^4)'"] & \bar{\mathcal{B}}\end{tikzcd}\tag{4.7}$$

We now explain the diagram (4.7). Let $\Sigma<\Sigma'$ be strata of the subspace (4.2). The space $\bar{\mathcal{B}}$ in the diagram (4.7) is defined analogously to the space $\bar{\mathcal{C}}_{\mathfrak{t}}$ defined in (2.11).

We now explain the maps $\rho^u_{\Sigma,\Sigma'}(S^4)$ and $\rho^d_{\Sigma,\Sigma'}(S^4)$ appearing in (4.7). The upper stratum, Σ', is given by a refinement of the partition giving the lower stratum, Σ. That is if $\Sigma<\Sigma'$ and Σ is given by the partition $\kappa_1+\cdots+\kappa_r$, then Σ' will be given by a partition $\kappa=\sum_{i,j}\kappa_{i,j}$ where $\kappa_i=\sum_j\kappa_{i,j}$. (There could be more than one such refinement. Each such refinement corresponds

to a different component of the end of Σ' near Σ and can be treated separately. For simplicity of exposition, we ignore both this issue and problems involving the symmetric group action here.) Let $\nu(\Sigma,\Sigma') \subset \Sigma'$ be the intersection of a tubular neighborhood of Σ in $\mathrm{Sym}^{\kappa,\diamond}(\mathbb{R}^4)$ with Σ'. The map $\rho^u_{\Sigma,\Sigma'}(S^4)$ in the diagram (4.7) is defined by the inclusion $\nu(\Sigma,\Sigma') \subset \Sigma'$.

Let $p_{\Sigma',\Sigma} : \nu(\Sigma,\Sigma') \to \Sigma$ be the tubular neighborhood projection map. The fiber of $p_{\Sigma',\Sigma}$ will be points in $\mathbb{R}^4$. Splice the connections given by a point in $\prod_j \bar{M}^{s,\diamond}_{spl,\kappa_{i,j}}(S^4)$ to the trivial connection at the points in $\mathbb{R}^4$ given by the point in the fiber of $p_{\Sigma',\Sigma}$. The resulting connection lies in $\bar{M}^{s,\diamond}_{spl,\kappa_i}(S^4)$ because of the inductive hypothesis that a neighborhood of the punctured trivial strata in $\bar{M}^{s,\diamond}_{spl,\kappa_i}(S^4)$ lies in the image of the splicing map with domain (4.6). The projection map $p_{\Sigma',\Sigma}$ and this splicing construction then define the map $\rho^d_{\Sigma,\Sigma'}(S^4)$ in the diagram (4.7).

Then, the associativity of splicing to the trivial connection (4.5) implies that the diagram (4.7) commutes. Specifically, the compositions

$$\boldsymbol{\gamma}_{\Sigma}(S^4)' \circ \rho^d_{\Sigma,\Sigma'}(S^4) \quad \text{and} \quad \boldsymbol{\gamma}_{\Sigma'}(S^4)' \circ \rho^u_{\Sigma,\Sigma'}(S^4)$$

are the splicing maps appearing on the left-hand-side and right-hand-side, respectively, of (4.5). Moreover, one can also show that any point in the images of both $\boldsymbol{\gamma}'_{\Sigma}$ and of $\boldsymbol{\gamma}'_{\Sigma'}$ appears in the pushout (4.7). Hence, the overlaps of the images of the splicing maps are then controlled by the pushout diagram (4.7).

Because the maps $\rho^u_{\Sigma,\Sigma'}(S^4)$ and $\rho^d_{\Sigma,\Sigma'}(S^4)$ are open embeddings, the union of the images of the splicing maps, W_κ, is a smoothly-stratified space. The gluing perturbation gives a smoothly-stratified isotopy between W_κ and $\mathcal{N}_\kappa$. The argument given before equation (4.4) then shows how to deform a collar of the boundary of $W_\kappa(S^4)$ into $\bar{M}^{s,\diamond}_{\kappa}(S^4)$. The resulting space is $\bar{M}^{s,\diamond}_{spl,\kappa}(S^4)$, completing the induction.

4.2. **Deforming the splicing map.** Let $\Sigma < \Sigma'$ be strata of $\mathrm{Sym}^{\ell}(X)$. With the domain, $\mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma)$, of the splicing maps $\boldsymbol{\gamma}'_{\Sigma}$ redefined by replacing the spaces $\bar{M}^{s,\diamond}_{\kappa}(S^4)$ appearing in the fiber of (3.2) with the spaces $\bar{M}^{s,\diamond}_{spl,\kappa}(S^4)$, we now describe the deformations of the splicing maps

$$\boldsymbol{\gamma}''_{\Sigma} : \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma) \to \bar{\mathcal{C}}_{\mathfrak{t}}, \quad \boldsymbol{\gamma}''_{\Sigma'} : \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma') \to \bar{\mathcal{C}}_{\mathfrak{t}}$$

needed to ensure that the overlap of their images is controlled by a pushout diagram of the form (4.1).

Let $\nu(\Sigma,\Sigma')$ again denote the intersection of Σ' with a tubular neighborhood of Σ. We define the space $\mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma,\Sigma')$ to be the restriction of the fiber bundle $\mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma')$ to $N_{\mathfrak{t}(\ell),\mathfrak{s}} \times \nu(\Sigma,\Sigma')$ appearing in (4.7) and we define

$\rho^u_{\Sigma,\Sigma'}$ to be the inclusion of bundles:

$$\begin{array}{ccc} \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma,\Sigma') & \xrightarrow{\rho^u_{\Sigma,\Sigma'}} & \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma') \\ \Pi_{\Sigma'}\downarrow & & \Pi_{\Sigma'}\downarrow \\ N_{\mathfrak{t}(\ell),\mathfrak{s}}\times\nu(\Sigma,\Sigma') & \longrightarrow & N_{\mathfrak{t}(\ell),\mathfrak{s}}\times\Sigma' \end{array} \tag{4.8}$$

The definition of

$$\rho^d_{\Sigma,\Sigma'}:\mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma,\Sigma')\to\mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma),$$

is similar to that of the map $\rho^d_{\Sigma,\Sigma'}(S^4)$ appearing in (4.7). Let $p_{\Sigma,\Sigma'}:\nu(\Sigma,\Sigma')\to\Sigma$ be the projection map of the tubular neighborhood. The fiber of $p_{\Sigma,\Sigma'}$ is, up to a choice of a trivialization of the tangent bundle of X, a collection of points in $\mathbb{R}^4$. The fiber of the composition,

$$(\mathrm{id}_{N_{\mathfrak{t}(\ell),\mathfrak{s}}}\times p_{\Sigma,\Sigma'})\circ\Pi_{\Sigma'}:\mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma,\Sigma')\to N_{\mathfrak{t}(\ell),\mathfrak{s}}\times\Sigma$$

is then a collection of points in $\mathbb{R}^4$ and the framed connections in the fiber of $\Pi_{\Sigma'}$. Just as in the construction of the map $\rho^d_{\Sigma,\Sigma'}(S^4)$, this data can be spliced to the trivial connection to get connections on S^4, giving an element of the fiber of Π_Σ. (Note that for the definition of the map $\rho^d_{\Sigma,\Sigma'}$ to make sense, we must redefine the fibers of Π_Σ in $\mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma)$ to have the connections in W_κ rather than $\mathcal{N}_\kappa$, as discussed in the beginning of §4.1.) This defines $\rho^d_{\Sigma,\Sigma'}$ as a fiber bundle map:

$$\begin{array}{ccc} \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma,\Sigma') & \xrightarrow{\rho^d_{\Sigma,\Sigma'}} & \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma) \\ \Pi_{\Sigma'}\downarrow & & \Pi_{\Sigma}\downarrow \\ N_{\mathfrak{t}(\ell),\mathfrak{s}}\times\nu(\Sigma,\Sigma') & \xrightarrow{\mathrm{id}_{N_{\mathfrak{t}(\ell),\mathfrak{s}}}\times p_{\Sigma,\Sigma'}} & N_{\mathfrak{t}(\ell),\mathfrak{s}}\times\Sigma \end{array}$$

Then, we wish to define deformations, $\boldsymbol{\gamma}''_\Sigma$ and $\boldsymbol{\gamma}''_{\Sigma'}$, of the splicing maps $\boldsymbol{\gamma}'_\Sigma$ and $\boldsymbol{\gamma}'_{\Sigma'}$, so that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma,\Sigma') & \xrightarrow{\rho^d_{\Sigma,\Sigma'}} & \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma) \\ \rho^u_{\Sigma,\Sigma'}\downarrow & & \gamma''_{\Sigma}\downarrow \\ \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma') & \xrightarrow{\gamma''_{\Sigma'}} & \bar{\mathcal{C}}_{\mathfrak{t}} \end{array} \tag{4.9}$$

The corresponding diagram for splicing to the trivial connection on S^4, (4.7), commuted because of the associativity of splicing equality, (4.5). No such equality holds when the connection to which one is splicing, A_0 in the language of §3.1, is not flat and the manifold to which one is splicing does not admit a flat metric. However, one can "flatten" the connection A_0 and the metric g on X on small balls around the splicing points in $\mathbf{x}$. To obtain the deformed splicing map, $\boldsymbol{\gamma}''_\Sigma$, one then simply "flattens" the connection

A_0 on larger neighborhoods of the splicing points, using a locally flattened metric to identify neighborhoods of these points with neighborhoods of the north pole in S^4. With this deformation of the splicing map, the diagram (4.9) commutes.

Define

$$\mathrm{Gl}(\mathfrak{t},\mathfrak{s},X) = \cup_{\Sigma\subset \mathrm{Sym}^\ell(X)} \gamma''_\Sigma(\mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma)). \tag{4.10}$$

The space $\mathrm{Gl}(\mathfrak{t},\mathfrak{s},X)$ is a smoothly-stratified space given by a union of local cone bundle neighborhoods. The pushout diagram (4.9) controls the overlaps of these diagrams and from this control, one can see that:

Theorem 4.1. *The space* $\mathrm{Gl}(\mathfrak{t},\mathfrak{s},X)$ *is a union of local cone bundle neighborhoods which satisfy the conditions of Definition 1.2.*

In [5], we will extend the results of [4] constructing a gluing perturbation of the image of these splicing maps to parameterize a neighborhood of $M_\mathfrak{s} \times \mathrm{Sym}^\ell(X)$ in $\bar{\mathcal{M}}_\mathfrak{t}/S^1$, giving the following technical result on which the proofs of Theorem 2.1 and Theorem 1.1 rely.

Theorem 4.2. *There is a section* $\mathfrak{o}$ *of a pseudo-vector bundle* $\Upsilon \to \mathrm{Gl}(\mathfrak{t},\mathfrak{s},X)$ *and a stratum-preserving deformation of the inclusion* $\mathrm{Gl}(\mathfrak{t},\mathfrak{s},X) \to \bar{\mathcal{C}}_\mathfrak{t}/S^1$ *such that the restriction of this deformation to* $\mathfrak{o}^{-1}(0)$ *parameterizes a neighborhood of* $M_\mathfrak{s} \times \mathrm{Sym}^\ell(X)$ *in* $\bar{\mathcal{M}}_\mathfrak{t}/S^1$. *In addition, the restriction of the obstruction section* $\mathfrak{o}$ *to any stratum vanishes transversely.*

Remark 4.3. The obstruction bundle, $\Upsilon \to \mathrm{Gl}(\mathfrak{t},\mathfrak{s},X)$, is only a pseudo-vector bundle because its rank depends on the stratum of $\mathrm{Gl}(\mathfrak{t},\mathfrak{s},X)$. Although Υ is not a vector bundle, using a relative Euler class argument, one can show that there is a rational cohomology class on $\partial\,\mathrm{Gl}(\mathfrak{t},\mathfrak{s},X)$ which acts as an Euler class of Υ. We refer to this class as $e(\Upsilon)$.

We omit any further discussion of the gluing perturbation because as it is a deformation, to do any cohomological calculations, it suffices to work with the images of the deformed splicing maps.

Finally, we observe that there is an S^1 action on the space $\mathrm{Gl}(\mathfrak{t},\mathfrak{s},X)$ and on the obstruction bundle Υ such that the splicing map and gluing perturbation are equivariant with respect to this action on $\mathrm{Gl}(\mathfrak{t},\mathfrak{s},X)$ and the action (2.7) on $\bar{\mathcal{C}}_\mathfrak{t}$.

5. The cohomological formalism

Using Theorem 4.2, the desired intersection number can be written as a cohomological pairing,

$$\#\left(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{n-1} \cap \mathbf{L}_{\mathfrak{t},\mathfrak{s}}\right) = \langle e(\Upsilon) \smile \bar{\mu}_p(z) \smile \bar{\mu}_c^{n-1}, [\partial\,\mathrm{Gl}(\mathfrak{t},\mathfrak{s},X)/S^1]\rangle, \tag{5.1}$$

to which Theorems 4.1 and 4.2 allow us to apply the formalism of §1.1. Explicitly, we write

$$\partial\,\mathrm{Gl}(\mathfrak{t},\mathfrak{s},X)/S^1 = \cup_i\, \mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i),$$

where $t_i : \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma_i) \to [0,\infty)$ is the cone function and

$$\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i) = \mathrm{Gl}(\mathfrak{t},\mathfrak{s},\Sigma_i)/S^1 \cap t_i^{-1}(\varepsilon_i) - \left(\cup_{j\neq i} t_j^{-1}([0,\varepsilon_j))\right).$$

The compatible structure group condition implies that there is a pre-compact subspace $K_i \Subset \Sigma_i$ such that the restriction of π_i to $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i)$ defines a fiber bundle,

$$\pi_i : \mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i) \to M_{\mathfrak{s}} \times K_i, \tag{5.2}$$

with the same structure group as the fiber bundle in (3.2). Hence, the characteristic classes of the bundle (5.2) are given by appropriate symmetric products of $p_1(X)$, $e(X)$, $c_1(\mathfrak{s})$, $c_1(\mathfrak{t})$, and $p_1(\mathfrak{t})$, and the cohomology class

$$\mu_{\mathfrak{s}} \in H^2(M_{\mathfrak{s}})$$

defining the Seiberg-Witten invariant.

As described after (1.5), the strata of $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i)$ are smooth manifolds with corners. The boundary of $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i)$ is

$$\partial\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i) = \cup_{j\neq i}\, \partial_j\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i) \quad \text{where} \quad \partial_j\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i) = \mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i)\cap t_j^{-1}(\varepsilon_j). \tag{5.3}$$

We abbreviate the cohomology class in (5.1) by

$$\Omega(z,n) = e(\Upsilon) \smile \bar\mu_p(z) \smile \bar\mu_c^{n-1}.$$

To apply the pushforward-pullback argument in §1.1 to compute the pairing (5.1), we need to select a representative of the cohomology class $\Omega(z,n)$ such that:

- The representative has compact support away from the boundaries $\partial\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i)$,
- The restriction of the representative to each component $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i)$ is a product of an equivariant cohomology class on the fiber and a cohomology class pulled back from the base of the fiber bundle (5.2).

We specify such a representative of $\Omega(z,n)$ by constructing a quotient, $\widehat{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$, of $\partial\,\mathrm{Gl}(\mathfrak{t},\mathfrak{s},X)/S^1$ from which $\Omega(z,n)$ is pulled back. Recall that $z = h^{\delta-2m}x^m$, where $h \in H_2(X;\mathbb{R})$ and $x \in H_0(X;\mathbb{Z})$ was the generator.

Proposition 5.1. *There is a quotient $\widehat{\mathbf{L}}^{vir}_{\mathfrak{t},\mathfrak{s}}$ of $\partial\,\mathrm{Gl}(\mathfrak{t},\mathfrak{s},X)/S^1$ with quotient map*

$$q : \partial\,\mathrm{Gl}(\mathfrak{t},\mathfrak{s},X)/S^1 \to \widehat{\mathbf{L}}^{vir}_{\mathfrak{t},\mathfrak{s}},$$

such that

$$\Omega(z,n) = q^*\widehat\Omega(z,n), \quad \text{for} \quad \widehat\Omega(z,n) \in H^d(\widehat{\mathbf{L}}^{vir}_{\mathfrak{t},\mathfrak{s}}), \tag{5.4}$$

where $d = \dim\partial\,\mathrm{Gl}(\mathfrak{t},\mathfrak{s},X)/S^1$, which has the following properties:

(1) *The image of each boundary, $q(\partial_j\mathbf{L}^{vir}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i))$, has codimension greater than or equal to two,*

(2) *The image of each component under the quotient map,*

$$\widehat{\mathbf{L}}^{vir}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i) = q(\mathbf{L}^{vir}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i)),$$

is a fiber bundle fitting into the diagram

$$\begin{array}{ccccc}
\mathbf{L}^{vir}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i) & \xrightarrow{q} & \widehat{\mathbf{L}}^{vir}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i) & \xrightarrow{\tilde{g}_i} & \mathrm{EG}_i \times_{G_i} \widehat{F}_i(\boldsymbol{\varepsilon}) \\
{\scriptstyle \pi_i}\downarrow & & {\scriptstyle \hat{\pi}_i}\downarrow & & {\scriptstyle m_i}\downarrow \\
M_{\mathfrak{s}} \times K_i & \longrightarrow & M_{\mathfrak{s}} \times \mathrm{cl}(\Sigma_i) & \xrightarrow{g_i} & \mathrm{BG}_i
\end{array} \tag{5.5}$$

where $\widehat{F}_i(\boldsymbol{\varepsilon})$ *is the fiber of* $\hat{\pi}_i$.

(3) *The homotopy type of the fiber bundle* $\hat{\pi}_i$ *in* (5.5) *depends only on the characteristic classes of the bundle* (5.2).

(4) *The restriction of* $\widehat{\Omega}(z,n)$ *to* $\widehat{\mathbf{L}}^{vir}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i)$, $\widehat{\Omega}_i$, *satisfies*

$$\widehat{\Omega}_i = \sum_j p_{i,j}(\tilde{g}_i^* \hat{\nu}^{d-j}) \smile (\widehat{\pi}_i^* \hat{\omega}_{i,j}), \tag{5.6}$$

where $\hat{\nu}$ *is the first Chern class of an* S^1 *action on the fibers* $\widehat{F}_i(\boldsymbol{\varepsilon})$, $\hat{\omega}_{i,j} \in H^j(M_{\mathfrak{s}} \times \widehat{K}_i)$ *is a polynomial in characteristic classes of* π_i *and the Poincaré duals of* h *and* x, *and* $p_{i,j}$ *are universal constants.*

Proof. We now sketch the construction of the quotient $\widehat{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}$. The quotient is defined by "collapsing" the boundaries of the components of the link defined in (5.3). From the compatible structure group conditions, we know that for $i<j$, the boundary $\partial_j \mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma)$ is a subbundle of the fiber bundle (5.2). From the Thom-Mather control conditions, we know that for $j<i$, the boundary $\partial_j \mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma)$ is the restriction of the bundle (5.2) to a boundary $M_{\mathfrak{s}} \times \partial_j K_i$ of the base.

We define the quotient of the boundary $\partial_j \mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma)$ for $j<i$. First, we observe that $\mathrm{cl}(\Sigma_i)$ can be presented as a quotient of K_i. Specifically, one notes that the union of singular strata in $\mathrm{cl}(\Sigma_i)$, $\mathrm{cl}(\Sigma_i)-\Sigma_i$, is a neighborhood deformation retraction in $\mathrm{cl}(\Sigma_i)$. The restriction of this retraction to $K_i \subset \Sigma_i$ can be used to define a surjective map $K_i \to \mathrm{cl}(\Sigma_i)$ which we consider as a quotient map. The fiber bundle (5.2) extends over $M_{\mathfrak{s}} \times \mathrm{cl}(\Sigma_i)$ and this extension can be presented as a quotient of $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma)$ which collapses the boundary $\partial_j \mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma)$ for $j<i$ to the restriction of the extended fiber bundle over the lower stratum $\Sigma_j \subset \mathrm{cl}(\Sigma_i)$.

The definition of the boundaries of $\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i)$ in (5.3) implies that

$$\partial_j \mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i) = \partial_i \mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_j).$$

Thus, for $i<j$ we must take the quotient of the boundary $\partial_j \mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i)$ exactly as we have done for the boundary $\partial_i \mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_j)$. Because the boundary $\partial_j \mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i)$ is a subbundle of the bundle (5.2), we can then use the property of compatible structure groups to argue that this quotient can be obtained

by taking a quotient of the fiber without changing the structure group. On the intersections,

$$\partial_{j_1}\partial_{j_2}\dots\partial_{j_r}\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i) = \cap_{k=1}^{r}\partial_{j_k}\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i) \quad \text{for } i < j_1 < \dots < j_r,$$

there are r quotients, defined by the inclusions

$$\partial_{j_1}\partial_{j_2}\dots\partial_{j_r}\partial_k\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i) \subset \partial_{j_k}\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i)$$

and the quotients on $\partial_{j_k}\mathbf{L}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i)$. One must verify that these multiple quotients are well-defined and respect the structure group.

The properties of the cohomology class $\Omega(z,n)$ are easily verified. □

Finally, we discuss how Theorem 2.1 follows from 5.1. The first condition in Proposition 5.1 yields the identity

$$q_*[\partial\,\mathrm{Gl}(\mathfrak{t},\mathfrak{s},X)/S^1] = \sum_i [\widehat{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i)]. \tag{5.7}$$

The third property in Proposition 5.1 and the pushforward-pullback formula imply that

$$(\hat{\pi}_i)_*(\tilde{g}_i^*\nu^k) = g_i^*(m_{i*}\hat{\nu}^k) \tag{5.8}$$

is given by a universal polynomial in the characteristic classes of $\hat{\pi}_i$. We then compute the intersection number in (5.1) by

$$\begin{aligned}
&\langle\Omega(z,n),[\partial\,\mathrm{Gl}(\mathfrak{t},\mathfrak{s},X)/S^1]\rangle\\
&\quad= \langle q^*\widehat{\Omega}(z,n),[\partial\,\mathrm{Gl}(\mathfrak{t},\mathfrak{s},X)/S^1]\rangle \quad \text{by (5.4)}\\
&\quad= \langle\widehat{\Omega}(z,n),q_*[\partial\,\mathrm{Gl}(\mathfrak{t},\mathfrak{s},X)/S^1]\rangle\\
&\quad= \sum_i \langle\widehat{\Omega}_i,[\widehat{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i)]\rangle \quad \text{by (5.7)}\\
&\quad= \sum_{i,j} p_{i,j}\langle\tilde{g}_i^*\hat{\nu}^{d-j}\smile\hat{\pi}_i^*\hat{\omega}_{i,j},[\widehat{\mathbf{L}}^{\mathrm{vir}}_{\mathfrak{t},\mathfrak{s}}(\Sigma_i)]\rangle \quad \text{by (5.6)}\\
&\quad= \sum_{i,j} p_{i,j}\langle(\hat{\pi}_i)_*(\tilde{g}_i^*\hat{\nu}^{d-j})\smile\hat{\omega}_{i,j},[M_{\mathfrak{s}}\times\mathrm{cl}(\Sigma_i)]\rangle\\
&\quad= \sum_{i,j} p_{i,j}\langle g_i^*(m_{i*}\hat{\nu}^{d-j})\smile\hat{\omega}_{i,j},[M_{\mathfrak{s}}\times\mathrm{cl}(\Sigma_i)]\rangle \quad \text{by (5.8)}
\end{aligned} \tag{5.9}$$

By the characterization of $g_i^*(m_{i*}\hat{\nu}^{d-j})$ following (5.8), we then see that the final expression is the desired universal polynomial in the characteristic classes appearing in Theorem 2.1.

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